

On the origin of multi-component anyon wave functions

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December 1992

Abstract

In this paper I discuss how the component structure of anyon wave functions arises in theories with non-relativistic matter coupled to a Chern-Simons gauge field on the torus. It is shown that there exists a singular gauge transformation which brings the Hamiltonian to free form. The gauge transformation removes a degree of freedom from the Hamiltonian. This degree of freedom generates only a finite dimensional Hilbert space and is responsible for the component structure of free anyon wave functions. This gives an understanding of the need for multiple component anyon wave functions from the point of view of Chern-Simons theory.

USITP-92-15

1 Introduction

Anyons, particles which obey fractional statistics [1, 2] (for a review see e.g. [3]), are now a well established phenomenon in theoretical physics. A controversy over whether arbitrary fractional statistics could be defined on compact surfaces has now been resolved by the work of several independent groups [4, 5, 6]. They find that anyon wave functions with arbitrary rational statistics parameter $\theta = \pi q/p$ can be defined on the torus if the wave functions have p components and N_A/κ is an integer (where N_A is the number of anyons). In general one cannot simultaneously diagonalize the operators which translate the anyons along the different cycles of the surface. However, one can pick a basis of wave functions so that the component index indicates the phase it picks up under translation along one cycle while translation along the other cycle shifts the component index by one unit.

In this paper I reconsider the problem of non-relativistic particles coupled to a U(1) Chern-Simons gauge field on the torus. I will consider a non-relativistic quantum field theory of bosons minimally coupled to the gauge field. The structure of the argument is as follows. Section two deals with the gauss constraint imposed by the Chern-Simons term and its solution on the torus. In section three the physical degrees of freedom of the gauge field are quantized followed by a discussion of the particle vacuum which is essentially a summary of an argument due to Polychronakos [11]. In section four the first quantized Hamiltonian is derived and the center of mass Hamiltonian is explicitly diagonalized. It is demonstrated in section five that only single component wave functions are needed in the full theory. Finally in section six it is shown that when the Hamiltonian is brought to free form by a singular gauge transformation the single component wave functions in the original theory become multiple component wave functions. It is argued that this is a consequence of removing a dynamical degree of freedom from the center of mass Hamiltonian which only generates a p dimensional Hilbert space.

Anyons on tori have been the subject of a number of recent investigations. In particular, discussions addressing similar issues to the ones presented in sections two and the first part of section four can be found in [6, 8, 17, 16] As mentioned earlier the discussion of the particle vacuum in section four is essentially a summary of arguments originally presented in [11].

Throughout this paper the torus will be taken to be the $L_1 \times L_2$ rectangle in the xy plane with opposite sides identified and the Chern-Simons coupling will be given by $\kappa = p/q$, where p and q

are relatively prime integers. The modular parameter τ appearing in the Jacobi theta functions is given by $\tau = iL_2/L_1$.

2 Solving the Chern-Simons constraint

This section will deal with the problem of solving the Gauss Law constraint imposed by the Chern-Simons term in the Lagrangian and isolating the remaining degrees of freedom of the gauge field which are to be quantized. On the torus the Gauss Law constraint does not completely determine the gauge potential, unlike the situation on the plane. This is because the Gauss Law only constrains the curvature associated with the C-S connection and does not completely specify the connection (up to gauge transformations). In fact, it is well known that to specify the connection completely on topologically non-trivial spaces one has to specify the Wilson lines $\exp i \oint a_\mu dx^\mu$ along all non-trivial loops [7]. In mathematical language this amounts to specifying the cohomology class of the connection. On the plane the de Rham cohomology group is trivial so that the curvature *does* determine the connection up to gauge transformation. This is not the case on the torus.

There is no entirely natural way of dividing the gauge field into real and constrained degrees of freedom. In particular the Wilson loops around the non-trivial cycles of the torus are affected by the amount of flux coming out of the torus and therefore depend implicitly on the constraint. To see that this is the case one can convince oneself that the flux flowing out of an area bounded by two loops along, say, the x-axis is given by the product of the Wilson loops evaluated along these two (oppositely oriented) loops. Therefore the Wilson loops must know about not only the flux flowing through the holes of the torus but also the flux flowing out of the torus. Nonetheless it is possible to divide the gauge field into real and constrained degrees of freedom by specifying a canonical solution to the constraint so that the connection is unambiguously solved for. Any terms which one can add to the connection while preserving the constraint will be real degrees of freedom and should be quantized. The remaining part of this section will be a concrete elaboration of this point.

Consider the following Lagrangian:

$$\begin{aligned}\mathcal{L} &= \frac{\kappa}{4\pi} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho + \psi^\dagger i D_0 \psi - \frac{1}{2m} (D_j \psi)^\dagger (D_j \psi) \\ D_0 &= \partial_0 - i a_0\end{aligned}$$

$$D_j = \partial_j - ia_j \quad (1)$$

Where a_μ is the Chern-Simons gauge field and the fields ψ are bosonic matter fields. From the above Lagrangian we get the constraint:

$$0 = \frac{\delta \mathcal{L}}{\delta a_0} = \frac{\kappa}{2\pi} f_{12} + J_0 \quad (2)$$

where

$$J_0 = \psi^\dagger \psi \quad (3)$$

It is important to realize that if the gauge field supports a non-zero total flux then there will be no globally well defined gauge potential on the torus. The mathematical reason for this is that the U(1) bundle over the torus is twisted. This can be seen by trying to calculate the total flux flowing out of the torus:

$$\begin{aligned} \phi &= \int \int dx dy (\partial_x a_y - \partial_y a_x) \\ &= \int_{y_0}^{y_0+L_2} dy (a_y(x_0+L_1, y) - a_y(x_0, y)) \end{aligned} \quad (4)$$

$$- \int_{x_0}^{x_0+L_1} dx (a_x(x, y_0+L_2) - a_x(x, y_0)) \quad (5)$$

If the gauge field were periodic along the two cycles of the torus the flux would vanish. If the gauge field does support flux and satisfies the following "quasi-periodic" boundary conditions:

$$a_j(\vec{r} + \hat{e}_i L_i) = a_j(\vec{r}) + \partial_j \Lambda_i \quad (6)$$

where the Λ_i are such that $\exp i\Lambda_i(x, y)$ are well defined (i.e. single-valued) gauge transformation on the torus¹. and if, in addition, the matter fields satisfy:

$$\psi(\vec{r} + \hat{e}_i L_i) = \exp(i\Lambda_i) \psi(\vec{r}) \quad (7)$$

then all gauge invariant observables such as the J_μ are well defined on the torus and there is no problem with consistency. I will work in the $\nabla \cdot \mathbf{a} = 0$ gauge, in which the Gauss law constraint

¹When there is an external electromagnetic field present the condition that $\exp i\Lambda_i(x, y)$ be single valued can be relaxed to requiring that $\exp i(\Lambda_i^{cs}(x, y) + \Lambda_i^{ext}(x, y))$ be single valued on the torus where $\Lambda_i^{ext}(x, y)$ is the corresponding gauge function of the external field.

can be written as:

$$\nabla^2 a_i = \frac{2\pi}{\kappa} \epsilon^{ij} \partial_j J_0 \quad (8)$$

It remains to give a canonical solution to the above equation and to identify the remaining degrees of freedom. I pick the following solution:

$$\begin{aligned} \hat{a}_1 &= \frac{2\pi}{\kappa} \left(\frac{yQ}{L_1 L_2} - \int \partial_2 G(\vec{r}' - \vec{r}) J_0 d^2 r' \right) \\ \hat{a}_2 &= \frac{2\pi}{\kappa} \int \partial_1 G(\vec{r}' - \vec{r}) J_0 d^2 r' \end{aligned} \quad (9)$$

where

$$Q = \int d^2 r J_0 \quad (10)$$

is the particle number operator. G is the periodic Green's function on the torus:

$$\nabla^2 G(r) = \delta(\vec{r}) - \frac{1}{L_1 L_2} \quad (11)$$

and is given by [8, 9]:

$$G(x, y) = \frac{1}{4\pi} \ln \frac{|\theta_1(z|\tau)|^2}{|\theta_1'(0|\tau)|^2} + \frac{y^2}{2L_2 L_1} \quad (12)$$

where $z = x + iy$ and θ_1 is the odd Jacobi theta function. The above solution fixes $\Lambda_1 = 0$ and $\Lambda_2 = 2\pi Qx/\kappa L_1$. It is easy to see that the only terms consistent with the constraint and the choice of transition functions (the Λ_i s) that one can add to a_i are position independent terms:

$$a_i = \frac{\theta_i}{L_i} + \hat{a}_i \quad (13)$$

In fact, only $\theta_i \bmod 2\pi$ is observable (the rest being gauge equivalent to 0)². The single valued transition functions respect the separation of the θ_i from the \hat{a}_i .

The flux flowing out of the torus ($2\pi Q/\kappa$) must be quantized according to the Dirac quantization condition which follows from requiring that the holonomy of any homotopically trivial closed path should be well defined. This imposes the condition that Q/κ be an integer. So the theory restricts the number of particles to be an integer multiple of p . This condition also makes the transition functions single valued on the torus.

²It should be clear from the context whether the θ_i stand for the Jacobi theta functions or the gauge degrees of freedom.

3 Quantization and the Structure of the Vacuum

I turn now to the quantization of the θ_i . The relevant term in the Lagrangian is:

$$\frac{\kappa}{4\pi} \epsilon^{ij} \int \dot{a}_i a_j d^{(2)}r = \frac{\kappa}{4\pi} \left(\dot{\theta}_1 \theta_2 - \dot{\theta}_2 \theta_1 - \frac{\pi}{\kappa} \left(\theta_2 \dot{Q} - \dot{\theta}_2 Q \right) + \int \left(\dot{\hat{a}}_1 \hat{a}_2 - \dot{\hat{a}}_2 \hat{a}_1 \right) \right) \quad (14)$$

By partial integration of the action one can remove all the time derivatives from θ_2 up to a total derivative term which contributes a surface term irrelevant to quantization. The variation of the action with respect to $\dot{\theta}_1$ gives the corresponding conjugate momentum ³:

$$\Pi \equiv \frac{\delta S}{\delta \dot{\theta}_1} = \frac{\kappa}{2\pi} \theta_2 \quad (15)$$

Imposing the canonical commutation relations gives:

$$[\theta_1, \Pi] = i = \frac{\kappa}{2\pi} [\theta_1, \theta_2] \quad (16)$$

Finally, it is necessary to construct a vacuum on which physical states can be built. The first quantized Hamiltonian will depend on the particle coordinates, momenta and the phases θ_i which are really global degrees of freedom (they are to be interpreted as the amount of flux flowing through the holes of the torus). A basis for the Hilbert space is provided by the set of states $\{|\theta_1\rangle | r_1 \dots r_n\rangle\}$.

Now try to construct a complete set of states in the θ sector. A natural first guess is to construct states in the ' θ_1 representation': $\{|\theta_1\rangle\}$, $\hat{\theta}_1 |\theta_1\rangle = \theta_1 |\theta_1\rangle$, $\hat{\theta}_2 |\theta_1\rangle = \frac{-i2\pi}{\kappa} \frac{\partial}{\partial \theta_1} |\theta_1\rangle$ with the completeness relation: $\int d\theta_1 |\theta_1\rangle \langle \theta_1| = 1$. Since the θ_i are not observables the summation overcounts by including physically equivalent states, states related by gauge transformations. The only observables which can be constructed from the θ_i are $U_i = \exp i\theta_i$ and therefore it is reasonable to construct a complete set of states with respect to these observables. The U_i satisfy

$$U_1 U_2 = U_2 U_1 \exp i \frac{2\pi}{\kappa} \quad (17)$$

$$U_1 U_2^p = U_2^p U_1 \quad (18)$$

Let $|\theta_1, \alpha_2\rangle$ represent a state on which

$$U_1 |\theta_1, \alpha_2\rangle = e^{i\theta_1} |\theta_1, \alpha_2\rangle \quad (19)$$

$$U_2^p |\theta_1, \alpha_2\rangle = e^{ip\alpha_2} |\theta_1, \alpha_2\rangle \quad (20)$$

³One may of course treat both θ_1 and θ_2 as coordinates but then one has to use Dirac brackets to quantize since the momentum conjugate to θ_2 vanishes identically. Of course both procedures give the same final result. I thank T. H. Hansson for pointing out an error in an earlier version of the manuscript concerning this point and bringing ref [10] to my attention where this question is discussed in its generality

The new completeness relation reads $\int_0^{2\pi} \int_0^{2\pi} d\alpha_1 d\alpha_2 \mid \alpha_1, \alpha_2 \rangle \langle \alpha_1, \alpha_2 \mid = 1$. Even this relation is not quite correct since the state $\mid \theta_1 + 2\pi, \alpha_2 \rangle$ does not lie on the same ray as $\mid \theta_1, \alpha_2 \rangle$ [11] even though they represent the same physical state. This is easily seen from the fact that the operators which perform these gauge transformations, $T_i = \exp\left(-2\pi \frac{\partial}{\partial \theta_i}\right)$, can be represented as $T_2 = \exp i\kappa\theta_1$ and $T_1 = \exp -i\kappa\theta_2$ and satisfy $T_1 T_2 = T_2 T_1 e^{-i2\pi\kappa}$. The irreducible representations of the T_i are q dimensional. The states $\mid \theta_1 + 2\pi l, \alpha_2 \rangle$, $l = 0, \dots, q-1$ are eigenstates of T_2 but transform into each other under the action of T_1 and do not lie on the same ray. Thus the full Hilbert space consists of q copies of the physical Hilbert space. Since the θ_i are being treated as phases, each physical sector of the Hilbert space just specifies a way of picking a branch for the phases and one is free to remain within one such sector. Indeed all physical observables commute with these transformations and thus restriction to one sector is equivalent to fixing a gauge.

Having constructed a basis of states in the θ_1 representation in the gauge field sector of the theory any eigenvector of θ_2 can be expressed as a linear superposition of these states. Which states does one need in order to construct the eigenstate of $\exp i\theta_2$ with eigenvalue $\exp i\delta$? To specify the state unambiguously one has to give the value of $\exp ip\theta_1$ as well. Consider the action of U_2 on the state $\mid \theta_1, \beta \rangle$. Since $\theta_2 = -i\frac{2\pi}{\kappa} \frac{\partial}{\partial \theta_1}$, the state $\mid \theta_1, \beta \rangle$ is mapped on to $\mid \theta_1 - \frac{2\pi}{\kappa}, \beta \rangle$ upto a phase factor. After p actions of U_2 the state returns to itself up to a phase (recalling that the states $\mid \theta_1 + 2\pi q, \beta \rangle$ and $\mid \theta_1, \beta \rangle$ lie on the same ray). Therefore $\exp i\theta_2$ can always be diagonalized by the p states $\{\mid \alpha_1 + 2\pi l/\kappa, \beta \rangle\}$, $l = 0 \dots p-1$, with possible eigenvalues $\beta + 2\pi n/\kappa$ where $\exp ip\alpha_1$ is the eigenvalue of U_1^p [11]. If δ belongs to the set $\{\beta + 2\pi n/\kappa\}$ then the corresponding eigenstate is a linear combination of the p states $\{\mid \alpha_1 + 2\pi l/\kappa, \beta \rangle\}$. Thus the gauge field Hilbert space is divided up into sectors labeled by the eigenvalues $U_1^p = e^{ip\alpha_1}$ and $U_2^p = e^{ip\alpha_2}$ and each sector is p dimensional [11]. With respect to the operators U_1 and U_2 the physical Hilbert space has the form $\mathcal{H} = \bigoplus_{\alpha_1, \alpha_2} \mathcal{H}_{\alpha_1, \alpha_2}$ where each $\mathcal{H}_{\alpha_1, \alpha_2}$ is p -dimensional. The particle vacuum state is then p -dimensional. This direct sum structure of the Hilbert space will turn out to be intimately related to the multi-component structure of anyon wave functions.

4 The Schrödinger Equation

In this section I will write down the Hamiltonian in first quantized form and define the Schrödinger wave functions. I will point out some new qualitative features in the Hamiltonian which distinguish

the torus from the plane. In particular I will concentrate on the center of mass Hamiltonian and argue that it contains essentially all the qualitatively new features of the Hamiltonian on the torus.

Following Jackiw and Pi [12], I take the Hamiltonian to be:

$$\mathcal{H} = -\frac{1}{2m}(D_i\psi)^\dagger D_i\psi \quad (21)$$

where the covariant derivatives have already been defined above. The wave functions are defined by:

$$\phi(\theta, x_i, y_i) = \langle \theta | \psi(r_1) \dots \psi(r_n) | \phi \rangle \quad (22)$$

To make the notation less cumbersome I have adopted the following abbreviated notation: θ stands for θ_1 , the eigenvalue of U_2^P is suppressed, and $|\theta\rangle$ is an abbreviation of $|\theta\rangle|0\rangle$, $|0\rangle$ being the particle vacuum.

The Schrödinger equation is then given by:

$$i\frac{\partial}{\partial t}\phi(\theta, x_i, y_i) = \langle \theta | [\psi(r_1) \dots \psi(r_n), H] | \phi \rangle \quad (23)$$

This determines the first quantized Hamiltonian, in the N_A particle sector, to be:

$$H = -\frac{1}{2m} \sum_{\alpha=1}^{N_A} \vec{D}_\alpha \cdot \vec{D}_\alpha \quad (24)$$

$$\vec{D}_\alpha = \vec{\nabla}_\alpha - i\vec{a}_\alpha \quad (25)$$

$$a_{\alpha x} = \frac{\theta}{L_1} + \frac{2\pi N_A}{\kappa L_1 L_2} y_\alpha + \frac{2\pi}{\kappa} \sum_{\beta \neq \alpha} \left(\frac{\partial}{\partial y_\alpha} G(x_\alpha - x_\beta, y_\alpha - y_\beta) \right) \quad (26)$$

$$a_{\alpha y} = -i\frac{2\pi}{\kappa} \frac{\partial}{\partial \theta} - \frac{2\pi}{\kappa} \sum_{\beta \neq \alpha} \left(\frac{\partial}{\partial x_\alpha} G(x_\alpha - x_\beta, y_\alpha - y_\beta) \right) \quad (27)$$

where G is the periodic Green's function given above. The expressions for the \vec{a}_α can be written in the simpler form:

$$a_{\alpha x} = \frac{\theta}{L_1} - \frac{2\pi N_A}{\kappa L_1 L_2} Y + \frac{i}{2\kappa} \frac{\partial}{\partial x_\alpha} \sum_{\beta \neq \alpha} \ln \frac{\theta_1^*(z_\alpha - z_\beta | \tau)}{\theta_1(z_\alpha - z_\beta | \tau)} \quad (28)$$

$$a_{\alpha y} = -i\frac{2\pi}{\kappa} \frac{\partial}{\partial \theta} + \frac{i}{2\kappa} \frac{\partial}{\partial y_\alpha} \sum_{\beta \neq \alpha} \ln \frac{\theta_1^*(z_\alpha - z_\beta | \tau)}{\theta_1(z_\alpha - z_\beta | \tau)} \quad (29)$$

where X and Y are the center of mass coordinates defined by $X \equiv \frac{1}{N_A} \sum_{\alpha=1}^{N_A} x_\alpha$ and $Y \equiv \frac{1}{N_A} \sum_{\alpha=1}^{N_A} y_\alpha$.

The Hamiltonian conveniently splits up into a center of mass plus a relative piece which commute with each other. The wave functions will then be of the product form $\psi^{cm} \otimes \psi^{rel}$ where each factor will satisfy the Schrödinger equation with respect to the appropriate Hamiltonian. It is illuminating to see the explicit form of the Hamiltonians:

$$H^{cm} = -\frac{1}{2mN_A} \left[\left(\frac{\partial}{\partial X} - i \frac{N_A \theta}{L_1} - i \frac{2\pi N_A^2}{\kappa L_1 L_2} Y \right)^2 + \left(\frac{\partial}{\partial Y} - \frac{2\pi N_A}{\kappa L_2} \frac{\partial}{\partial \theta} \right)^2 \right] \quad (30)$$

$$H^{rel} = -\frac{1}{2mN_A} \sum_{\alpha, \beta, \alpha \neq \beta} \left[\left(\frac{\partial}{\partial x_\alpha} - \frac{\partial}{\partial x_\beta} - \frac{1}{2\kappa} \left(\frac{\partial}{\partial x_\alpha} - \frac{\partial}{\partial x_\beta} \right) \Lambda \right)^2 + \left(\frac{\partial}{\partial y_\alpha} - \frac{\partial}{\partial y_\beta} - \frac{1}{2\kappa} \left(\frac{\partial}{\partial y_\alpha} - \frac{\partial}{\partial y_\beta} \right) \Lambda \right)^2 \right] \quad (31)$$

where

$$\Lambda = \sum_{\mu < \nu} \ln \frac{\theta_1^*(z_\mu - z_\nu | \tau)}{\theta_1(z_\mu - z_\nu | \tau)} \quad (32)$$

H^{rel} can be understood as the generalization of the Hamiltonian for the relative coordinates on the plane. It is well known that the $\theta_1(z | \tau)$ are the torus analogs of $z = x + iy$ on the plane [13], and therefore the expression $\ln \frac{\theta_1^*(z_k - z_l | \tau)}{\theta_1(z_k - z_l | \tau)}$ corresponds to $\ln \frac{z^*}{z}$ on the plane. The Hamiltonian is mapped to the free Hamiltonian by an obvious transformation analogous to the one on the plane. As far as H^{rel} is concerned everything is analogous to the case on the plane.

The center of mass Hamiltonian, on the other hand, is quite a different object and there is no simple analogy between it and the corresponding Hamiltonian on the plane. On the plane the center of mass Hamiltonian is explicitly free and knows nothing about the flux tubes attached to the particles. On the torus, however, the Hamiltonian is not free, but, as I will show, there is a transformation which takes it to a free form. For such a transformation to exist it will turn out to be necessary that the θ_i be quantized. This crucial difference will be responsible for the component structure of the anyon wave functions on the torus. In fact, I will show that if the θ_i are not quantized the picture of anyons as interacting Aharonov-Bohm tubes breaks down.

Since I am only interested in revealing the component structure of anyon wave functions and not in finding exact solutions for the entire Hamiltonian, I will restrict my attention to the center of mass Hamiltonian in the following. In the previous section I discussed the non-trivial behavior

of the gauge field sector under the transformations $\theta_i \rightarrow \theta_i + 2\pi$. In particular it was shown that associated with each physical value of $\exp i\theta_1$ there were q linearly independent Hilbert space rays any of which could be reached from any other by the action of T_1 an appropriate number of times. The second quantized Hamiltonian is invariant under the combined transformations $T_j : \theta_j \rightarrow \theta_j + 2\pi$ and $\psi \rightarrow \exp\left(\frac{i2\pi x_j}{L_j}\right) \psi$. This invariance is reflected in the first quantized theory by the set of conditions:

$$T_1 \phi(X, Y, \theta) = e^{i\gamma} \exp\left(i \frac{2\pi X}{L_1}\right) \phi(X, Y, \theta) \quad (33)$$

$$T_2 \phi(X, Y, \theta) = e^{i\beta} \exp\left(i \frac{2\pi Y}{L_2}\right) \phi(X, Y, \theta) \quad (34)$$

Due to the non-commutativity of the operators which translate the θ_i it is not possible to simultaneously impose the above conditions. Instead, the most general conditions one may impose consistent with the commutation relations of the T_i are:

$$T_1 \phi_l(X, Y, \theta) = e^{i\gamma} \exp\left(i \frac{2\pi X N_A}{L_1}\right) \phi_{l-1}(X, Y, \theta) \quad (35)$$

$$T_2 \phi_l(X, Y, \theta) = e^{-i\beta - i2\pi\kappa l} \exp\left(i \frac{2\pi Y N_A}{L_2}\right) \phi_l(X, Y, \theta) \quad (36)$$

The second condition states that

$$\exp -i(\kappa\theta + 2\pi N_A Y/L_2) \phi_l(X, Y, \theta) = \exp -i(\beta + 2\pi\kappa l) \phi_l(X, Y, \theta) \quad (37)$$

The first condition requires that

$$\phi_l(X, Y, \theta + 2\pi) = e^{i\gamma} \exp\left(i \frac{2\pi X N_A}{L_1}\right) \phi_{l-1}(X, Y, \theta) \quad (38)$$

Turning now to the Hamiltonian, note that:

$$\left[\left(\frac{\partial}{\partial Y} - \frac{2\pi N_A}{\kappa L_2} \frac{\partial}{\partial \theta} \right), \left(\frac{N_A}{L_1} \theta + \frac{2\pi N_A^2}{\kappa L_1 L_2} Y \right) \right] = 0 \quad (39)$$

This tells us that the operators ∂_X , $\left(\frac{N_A}{L_1} \theta + \frac{2\pi N_A^2}{\kappa L_1 L_2} Y \right)$, and $\left(\partial_Y - \frac{2\pi N_A}{\kappa L_2} \frac{\partial}{\partial \theta} \right)$ can be diagonalized simultaneously. The situation is complicated by the fact that the Hamiltonian is not periodic on the torus (because of the presence of a non-zero flux) making it necessary to impose boundary conditions which do not respect the commutativity of these operators. In particular, one has to sum over eigenstates of the Hamiltonian which carry distinct eigenvalues of the operator $P_X = -i\partial_X$

which makes it impossible to diagonalize P_X (this is eloquently explained in the appendix in [14]). What is important, though, is that one can immediately write down the eigenstates of the Hamiltonian from which one can construct solutions obeying the correct boundary conditions.

$$\chi_l^{\vec{k}\vec{m}}(X, Y, \theta) = e^{i(2\pi k_1 X/L_1)} e^{i(2\pi k_2 Y/L_2)} e^{i(\frac{\gamma}{2\pi} + m_2)\theta} \delta\left(\kappa\theta + \frac{2\pi Y N_A}{L_2} - \beta - 2\pi\kappa l - 2\pi m_1\right) \quad (40)$$

are eigenstates of the Hamiltonian but do not satisfy the correct quasi-periodic boundary conditions in Y and θ . That is the wave function does not satisfy equation (38) and the quasi-periodicity condition in Y :

$$\phi(X, Y + L_2, \theta) = \exp\left(\frac{i2\pi N_A^2 X}{\kappa L_1}\right) \phi(X, Y, \theta). \quad (41)$$

The argument of the delta function in χ has been chosen so as to satisfy equation (37). It is a simple exercise to show that the correct combination of these eigenstates is:

$$\begin{aligned} \Phi_{ln}^{\vec{k}\vec{m}}(X, Y, \theta) &= \sum_{j=-\infty}^{\infty} \chi^{k_1 + N_A l + \frac{N_A}{\kappa}(pj+n), k_2, m_2, (jp+n+m_1)} \\ &= e^{i\frac{2\pi Y}{L_2}\left(k_2 - \frac{N_A}{\kappa}(m_2 + \gamma/2\pi)\right)} e^{i(\gamma/2\pi + m_2)(\theta + 2\pi N_A Y/\kappa L_2)} \\ &\quad \sum_{j=-\infty}^{\infty} e^{i\frac{2\pi X}{L_1}\left(k_1 + lN_A + \frac{N_A}{\kappa}(pj+n)\right)} \delta\left(\kappa\theta + \frac{2\pi Y N_A}{L_2} - \beta - 2\pi\kappa l - 2\pi(jp + n + m_1)\right) \end{aligned} \quad (42)$$

The index $n = 0, \dots, p-1$ is a degeneracy index. Note that these wave functions are non-zero for only a discrete set of values of $\kappa\theta + 2\pi N_A Y/L_2$. They are eigenstates of the Hamiltonian with eigenvalues:

$$E_{\vec{k}\vec{m}} = \frac{1}{2mN_A} \left(\left(\frac{2\pi}{L_1} k_1 - \frac{N_A}{\kappa L_1} (\beta + 2\pi m_1) \right)^2 + \left(\frac{2\pi}{L_2} k_2 - \frac{N_A}{\kappa L_2} (\gamma + 2\pi m_2) \right)^2 \right) \quad (43)$$

The energy is independent of the gauge index l as one would like, and most importantly it is of the free particle form. Interestingly, β and γ , the global phases allowed in the most general boundary conditions for θ , shift the energy spectrum.

5 Single Valued Wave Functions on the Torus

In this section I will show that it is possible to define single valued wave functions on the torus. It was shown above that the wave function must pick up a gauge transformation after translation around at least one cycle. Therefore, it is necessary to specify what one means by single valued wave

functions in this context. The wave function should really be thought of as a section on the $U(1)$ -bundle over the torus. So a well defined wave function on the torus should transform as a section which is completely specified once one has picked a set of transition functions. Alternatively, one may pick the translation operators to be the following:

$$t_\alpha(L_i \hat{e}_i) = \exp(-i\Lambda_i) \exp(L_i \partial / \partial x_{\alpha i}) \quad (44)$$

under the action of which the wave function should be single valued. The group of such translations is known in the condensed matter literature as the magnetic translation group. It is important to note that in the case considered here the generators of magnetic translations $P_{\alpha i} = (\partial / \partial x_{\alpha i} - i\Lambda_i / L_i)$ do not commute with the Hamiltonian, but finite translations by lattice vectors do commute with the Hamiltonian. The situation is, therefore, somewhat different from the case of a constant magnetic field where this group usually appears.

This group of translations gives a precise way of formulating what one means by single valued wave functions on the torus: that the wave functions should be single valued under the group of magnetic translations. The magnetic translation group has been studied extensively in the condensed matter literature. Haldane has studied this group for the many body case [15]. I will follow Haldane's analysis, providing slightly more detail where necessary, and show that there is no obstruction to defining single valued wave functions on the torus. First, define the following operators:

$$t_\alpha(\vec{a}_i) = \exp \vec{P}_\alpha \cdot \vec{a} \quad (45)$$

and note that

$$t_\alpha(\vec{a}) t_\alpha(\vec{b}) = t_\alpha(\vec{b}) t_\alpha(\vec{a}) \exp \left(-i \frac{2\pi N_A}{\kappa L_1 L_2} (\vec{a} \times \vec{b}) \right) \quad (46)$$

When \vec{a} and \vec{b} are restricted to be of the form $\vec{L}_{mn} = m\vec{L}_1 + n\vec{L}_2$, where m, n are integers, the translation operators above commute among themselves and with the Hamiltonian and may be simultaneously diagonalized. To see how one diagonalizes these operators it is convenient to factorize them into a center of mass translation T and a relative translation \tilde{t} as follows:

$$t_\alpha(\vec{L}_{mn}) = T \left(\frac{\vec{L}_{mn}}{N_A} \right) \tilde{t}_\alpha(\vec{L}_{mn}) \quad (47)$$

where

$$T(\vec{a}) = \prod_{\alpha} t_{\alpha}(\vec{a})$$

$$\tilde{t}_{\alpha}(\vec{a}) = T\left(-\frac{\vec{a}}{N_A}\right) t_{\alpha}\left(\frac{\vec{a}}{N_A}\right) \quad (48)$$

The center of mass translation operator $T(\vec{a})$ and the relative translation operators $\{ \tilde{t}_{\alpha}(\vec{b}) \}$ commute for arbitrary arguments.

From the above commutation relations it is readily established that:

$$\tilde{t}_{\alpha}(\vec{L}_1) \tilde{t}_{\beta}(\vec{L}_2) = \tilde{t}_{\beta}(\vec{L}_2) \tilde{t}_{\alpha}(\vec{L}_1) \exp i \frac{2\pi}{\kappa}$$

$$T\left(\frac{\vec{L}_1}{N_A}\right) T\left(\frac{\vec{L}_2}{N_A}\right) = T\left(\frac{\vec{L}_2}{N_A}\right) T\left(\frac{\vec{L}_1}{N_A}\right) \exp -i \frac{2\pi}{\kappa} \quad (49)$$

These commutation relations imply that it is not possible to write down a single valued wave function of the form $\psi^{cm} \otimes \psi^{rel}$. However, it is still possible to write a single valued wave function of the form $\sum_i \psi_i^{cm} \otimes \psi_i^{rel}$. To establish this result I will now diagonalize a maximal subset of translation operators in each sector and find the single valued combination.

The largest commuting set of center of mass translation operators is, $T\left(p\frac{\vec{L}_2}{N_A}\right)$, $T\left(\frac{\vec{L}_1}{N_A}\right)$. On the set of functions I defined in the previous section their action is:

$$T\left(\frac{\vec{L}_1}{N_A}\right) \phi_{ln}^{\vec{k}\vec{m}} = e^{i2\pi k_1/N_A} e^{i2\pi n/\kappa} \phi_{ln}^{\vec{k}\vec{m}} \quad (50)$$

$$T\left(\frac{\vec{L}_2}{N_A}\right) \phi_{ln}^{\vec{k}\vec{m}} = e^{i2\pi k_2/N_A} \phi_{l,n-1}^{\vec{k}\vec{m}}$$

$$T\left(p\frac{\vec{L}_2}{N_A}\right) \phi_{ln}^{\vec{k}\vec{m}} = e^{i2\pi p k_2/N_A} \phi_{ln}^{\vec{k}\vec{m}} \quad (51)$$

Similarly for the relative coordinate wave functions the maximal commuting set of magnetic translation operators are $\tilde{t}_{\alpha}(\vec{L}_1)$, $\tilde{t}_{\alpha}(p\vec{L}_2)$. Now let $\psi_{\zeta_s}^{\vec{k}}$ stand for an eigenstate of the relative coordinate Hamiltonian satisfying the following:

$$\tilde{t}_{\alpha}(\vec{L}_1) \psi_{\zeta_n}^{\vec{k}} = e^{-i2\pi k_1/N_A} e^{-i2\pi n/\kappa} \psi_{\zeta_n}^{\vec{k}}$$

$$\tilde{t}_{\alpha}(p\vec{L}_2) \psi_{\zeta_n}^{\vec{k}} = e^{-i2\pi p k_2/N_A} \psi_{\zeta_n}^{\vec{k}} \quad (52)$$

Then by the commutation relations:

$$\tilde{t}_\alpha \left(\vec{L}_2 \right) \psi_{\zeta n}^{\vec{k}} = e^{-i2\pi k_2/N_A} \psi_{\zeta, n-1}^{\vec{k}} \quad (53)$$

The index ζ keeps track of any other quantum numbers which the relative wave function may carry.

The above classification of the center of mass and relative wave functions in terms of magnetic translation eigenstates allows one to immediately write down the total single valued wave function:

$$\Psi_{l\zeta}^{\vec{k}\vec{m}} = \sum_{n=1}^p \phi_{ln}^{\vec{k}\vec{m}} \psi_{\zeta n}^{\vec{k}} \quad (54)$$

Notice that I have *not* shown that for any vector \vec{k} there exists a state $\psi_{\zeta n}^{\vec{k}}$ satisfying the above conditions. One should really think of the quantum numbers \vec{k} as being provided by the relative wave function which (possibly) restrict the allowed values of \vec{k} appearing in the center of mass wave function to keep the total wave function single valued.

It is important to note that there is no component structure when the unconstrained part of the Chern-Simons gauge field is present in the Hamiltonian. This is in contrast to what is claimed in ref [17] where a different Schrödinger wave function is defined.

6 Component Structure of free Anyon Wave Functions

It has long been known that for anyons on the plane one can adopt one of two pictures [1]. One works either with explicitly "anyonic" (multivalued) wave functions or with single valued wavefunctions but with Aharonov-Bohm flux tubes attached to the particles. The equivalence of the two pictures is established by constructing a unitary transformation which maps the single valued wave functions to anyonic wave functions satisfying the Schrödinger equation with respect to a Hamiltonian without flux tubes.

The Chern-Simons term is often added as a convenient way of attaching flux tubes to the particles. Indeed, on the plane, the constraint determines the gauge field completely and the C-S term's sole purpose is to attach these flux tubes to the particles. As is well known and shown above, on the torus the C-S term does more than just attach flux tubes: it also quantizes the flux quanta flowing through the holes of the torus. It may appear that the intuitive picture of anyons as Aharonov-Bohm tubes is a full description of anyons even on the torus and therefore that the

presence of a Chern-Simons term yields a non-minimal description of anyons. I would like to show that this is not the case, that one *must* quantize the θ_i for the single valued wave functions to be related by a unitary transformation to the anyonic wave functions.

The negative result that when the θ_i are *not* quantized there is no unitary transformation relating the single valued wave functions to free anyon wave functions follows from a straightforward argument. First suppose that the constraint $f_{12} = -\frac{2\pi}{\kappa} J_0$ is given but that it is not generated by a Chern-Simons term. One would proceed as before only now, due to the absence of the C-S term, the θ_i are not quantized but are c-numbers. The center of mass Hamiltonian is then given by

$$H^{cm} = -\frac{1}{2mN_A} \left[\left(\frac{\partial}{\partial X} - i\frac{N_A\theta_1}{L_1} - i\frac{2\pi N_A^2}{\kappa L_1 L_2} Y \right)^2 + \left(\frac{\partial}{\partial Y} - i\frac{N_A\theta_2}{L_2} \right)^2 \right] \quad (55)$$

But this is just the Hamiltonian for a particle of mass mN_A in a constant magnetic field of strength $b = -\frac{2\pi N_A^2}{\kappa L_1 L_2}$. There is no unitary transformation which takes this Hamiltonian to a free form. Moreover, the energy spectrum is of the form $E = \omega \left(n + \frac{1}{2} \right)$ which is not at all of the free form established above. Therefore, pure Aharonov-Bohm flux tubes is not a description of anyons on the torus.

The positive result that there is a unitary equivalence between free anyons and particles interacting with a Chern-Simons gauge field on the torus remains to be established. To construct the transformation I remind the reader that

$$\left[\left(\frac{\partial}{\partial Y} - \frac{2\pi N_A}{\kappa L_2} \frac{\partial}{\partial \theta} \right), \left(\frac{N_A}{L_1} \theta + \frac{2\pi N_A^2}{\kappa L_1 L_2} Y \right) \right] = 0 \quad (56)$$

Define a new center of mass wave function $\tilde{\phi}_{ln}^{\vec{k}\vec{m}}$ by:

$$\phi_{ln}^{\vec{k}\vec{m}} = \exp \left[i \frac{N_A X}{L_1} \left(\theta + \frac{2\pi N_A}{\kappa L_2} Y \right) \right] \tilde{\phi}_{ln}^{\vec{k}\vec{m}} \quad (57)$$

$\tilde{\phi}_{ln}^{\vec{k}\vec{m}}$ satisfies the Schrödinger equation for a free particle of mass mN_A . The "gauge" transformed center of mass Hamiltonian depends only on the conjugate momenta $-i\partial_X$ and $-i\partial_u = -i\partial_Y + i\frac{2\pi N_A}{\kappa L_2} \partial_\theta$ but not on the coordinates X and $u=Y$; neither does it depend on the coordinate $v = \theta + \frac{2\pi N_A}{\kappa L_2} Y$ nor its conjugate momentum $-i\partial_v = -i\partial_\theta$. Hence the Hamiltonian is free in X and u and is independent of v and $-i\partial_v$. From the form of the transformed Hamiltonian one would conclude that v is a constrained variable and its Hamiltonian evolution is trivial, allowing arbitrary functions of v to be perfectly good wave functions. However, this is not the case, since the

complicated boundary conditions that the wave function must satisfy restrict its form considerably as I will show by looking at the explicit form of the $\tilde{\phi}$.

H^{rel} is brought to free form by writing

$$\psi_{\zeta n}^{\vec{k}} = \prod_{\alpha < \beta} \left(\frac{\theta_1^*(z_\alpha - z_\beta | \tau)}{\theta_1(z_\alpha - z_\beta | \tau)} \right)^{\frac{1}{2\kappa}} \tilde{\psi}_{\zeta n}^{\vec{k}} \quad (58)$$

The wave function $\tilde{\psi}_{\zeta n}^{\vec{k}}$ solves the Schrödinger equation for the free Hamiltonian in the relative coordinates.

Note that both gauge transformations are "singular". For the gauge transformation on the relative coordinates it is well known that there is no smooth well defined extrapolation to the points where the particle coordinates coincide (the "origins" in the relative coordinates). The center of mass gauge transformation (57) is singular in the sense that if one embeds the torus in a higher dimensional space (e.g. in R^3) there is no smooth well defined extrapolation to the entire space. This just means that at some point there is a source for the flux and there is no way of gauging the source away. But neither one of these singularities is particularly troublesome since in the first case the wave function vanishes identically whenever coordinates for any two particles coincide, and in the second one is always working on the surface of the embedded solid torus.

The transformed total wave function is

$$\Psi_{l\zeta}^{\vec{k}\vec{m}} = \exp \left[i \frac{N_A X}{L_1} \left(\theta + \frac{2\pi N_A}{\kappa L_2} Y \right) \right] \prod_{\alpha < \beta} \left(\frac{\theta_1^*(z_\alpha - z_\beta | \tau)}{\theta_1(z_\alpha - z_\beta | \tau)} \right)^{\frac{1}{2\kappa}} \tilde{\Psi}_{l\zeta 0}^{\vec{k}} \quad (59)$$

The index 0 is included in anticipation of the component structure to be revealed soon. In terms of the center of mass and relative coordinates the total free wave function is given by:

$$\tilde{\Psi}_{l\zeta 0}^{\vec{k}\vec{m}} = \sum_{n=1}^p \tilde{\phi}_{ln}^{\vec{k}\vec{m}} \tilde{\psi}_{\zeta n}^{\vec{k}} \quad (60)$$

Now I turn to the behavior of this wave function under translation by a lattice vector. First, define the following set of wave functions:

$$\tilde{\Psi}_{l\zeta j}^{\vec{k}\vec{m}} = \sum_{n=1}^p e^{-i \frac{2\pi n j}{\kappa}} \tilde{\phi}_{ln}^{\vec{k}\vec{m}} \tilde{\psi}_{\zeta n}^{\vec{k}} \quad (61)$$

For $j = 0$ the original wave function is recovered. When one translates particle α by a lattice vector in the x -direction, $x_\alpha \rightarrow x_\alpha + L_1$, while keeping all others fixed, the wave function changes

in the following way:

$$\tilde{\Psi}_{l\zeta j}^{\vec{k}\vec{m}}(x_\alpha + L_1) = \left(e^{-i\pi/\kappa}\right)^{N_A-1} e^{-i(\beta/\kappa+2\pi m_1/\kappa)} \tilde{\Psi}_{l\zeta, j+1}^{\vec{k}\vec{m}}(x_\alpha) \quad (62)$$

Thus p translations of any particle in the x direction generates the entire set of linearly independent wave functions defined above. When one translates particle α by a lattice vector in the y -direction the above wave functions transform as follows:

$$\tilde{\Psi}_{l\zeta j}^{\vec{k}\vec{m}}(y_\alpha + L_2) = \left(e^{i\pi/\kappa}\right)^{N_A-1} e^{-i2\pi j/\kappa} \tilde{\Psi}_{l\zeta, j}^{\vec{k}\vec{m}}(y_\alpha) \quad (63)$$

This shows that the transformed wave function has p -components.

One view of the component structure is provided by looking at the singular gauge transformation on the center of mass wave function:

$$\exp\left[-i\frac{N_A X}{L_1}\left(\theta + \frac{2\pi N_A Y}{L_2}\right)\right] \quad (64)$$

Since the total wave function is not an eigenstate of $\exp i\left(\theta + \frac{2\pi N_A Y}{L_2}\right)$, the wave function can not pick up a total phase under translations $X \rightarrow X + L_1/N_A$ but is instead transformed to an entirely different wave function. This is a reflection of the fact that the translation group was represented projectively and to make the action of the translation group on the total wave function single component and single valued one had to sum over distinct eigenstates of $\exp i v$. From this point of view the component structure arises due to the fact that translation symmetry is realized projectively in the original problem forcing one to abandon the $\psi^{cm} \otimes \psi^{rel}$ form of the wave functions in favor of single component and single valued wave functions which do not have a simple product form and are hence not eigenstates of $\exp i v$.

Another approach to understanding this phenomenon is provided by working in a different basis.

$$\tilde{\Phi}_{l\zeta n}^{\vec{k}\vec{m}} = \tilde{\phi}_{ln}^{\vec{k}\vec{m}} \tilde{\psi}_{\zeta n}^{\vec{k}} \quad (65)$$

These wave functions are eigenstates of translations in the x -direction but not in the y -direction. They can be written in a factorized form:

$$\begin{aligned} \tilde{\Phi}_{l\zeta n}^{\vec{k}\vec{m}} &= \left[e^{i\frac{2\pi Y}{L_2}(k_2 - N_A(\gamma + 2\pi m_2)/\kappa)} e^{i\frac{2\pi X}{L_1}(k_1 - N_A(\beta + 2\pi m_1)/\kappa + \frac{N_A}{\kappa}n)} \right] e^{-i\frac{2\pi N_A X n}{\kappa L_1}} \\ &\quad \left[\sum_{j=-\infty}^{\infty} \exp i(\gamma/2\pi + m_2)(\beta/\kappa + 2\pi l + 2\pi/\kappa(jp + n + m_1)) \right] \end{aligned}$$

$$\delta \left(\kappa \theta + \frac{2\pi Y N_A}{L_2} - \beta - 2\pi \kappa l - 2\pi (jp + n + m_1) \right) \tilde{\psi}_{\zeta_n}^{\vec{k}} \quad (66)$$

The second factor in square brackets depends only on $v = \theta + 2\pi N_A Y / \kappa L_2$, and is not periodic under translations in the y direction. In fact, it is this part of the wave function which is responsible for the component structure. Wave functions in v have trivial Hamiltonian evolution (since the transformed Hamiltonian is independent of v and its conjugate momentum). The wave functions in v written above are exactly those found by Polychronakos [11] for pure Chern-Simons with θ replaced by v and generalized to cases where $q \neq 1$. Usually a variable with respect to which the Hamiltonian vanishes is allowed to have arbitrary wave functions. However, as Polychronakos showed for the case of pure Chern-Simons, when one has a compact phase space the Hilbert space is finite dimensional. This is exactly what has happened here ⁴. The Hilbert space of wave functions in v is p -dimensional. Thus the component structure really comes about due to the appearance of a variable which generates a finite (p) dimensional Hilbert space of wave functions which have trivial Hamiltonian evolution. This variable is a combination of particle and gauge degrees of freedom of such a form that the wave functions turn out not to be periodic under lattice translations. So in this second view the component structure arises due to a peculiar mixing of gauge field and particle degrees of freedom which do not allow one to have a single component wave function. These wave functions are completely analogous to the ones which appear for the particle vacuum (the pure Chern-Simons case).

From the point of view of Chern-Simons theory what is surprising is not that the anyon wave functions have multiple components but that they only have a finite number of components. The reason is that one starts out with $2N_A + 1$ coordinates but in the end one has only $2N_A$ coordinates with non-trivial Hamiltonian evolution. One would expect that the degree of freedom which has been "gauged" away should generate an infinite degeneracy of which some infinite set is related by lattice translations. Instead one finds that this degree of freedom generates only a finite dimensional Hilbert space! This is a very surprising result indeed.

I close with a comment. The gauge transformation used above to get anyon wave functions is not the most general. One may always supplement the transformation with an extra factor $\exp iav$ where a is arbitrary. Since a and $a+1$ are related by an x -translation both transformations project to members of the same p dimensional space of anyon wave functions. Therefore one may

⁴Taking $u = Y$ gives $-i\partial_v = -i\partial_\theta = \frac{\kappa}{2\pi}\theta_2$. Clearly v and p_v generate a compact phase space (θ_2 and $\theta_2 + 2\pi$ are identified)

always restrict $0 \leq a < 1$, this gives us the entire one parameter family of possible anyon wave functions. The parameter a just shifts the eigenvalues of y -translation by a constant amount. The correspondence between free anyons to bosons coupled to a Chern-Simons gauge field is therefore many to one.

7 Acknowledgments

I am indebted to T. H. Hansson, A. Karlhede and M. Ljungberg for discussions while this work was in progress and for reading the manuscript critically. I am also indebted to Marcelo Camperi for introducing me to anyons and Chern-Simons theories and bringing ref [6] to my attention. I thank F. Bastianelli and E. Westerberg for stimulating discussions.

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